WS 8.3

1 Problem 1

1. Calculate $\int_0^4 \sqrt{16-x^2}\,dx$.

We'll do a trig sub of $x = a\sin u$ (so $dx = a\cos u\,du$) since our integral contains $\sqrt{a^2-x^2}$ with $a=4$:

$$\begin{aligned}
\int_0^4 \sqrt{16-x^2}\,dx = \int_{u=\sin^{-1}(0/4)}^{u=\sin^{-1}(4/4)} \sqrt{16-(4\sin u)^2}(4\cos u\,du) &= \int_0^{\pi/2} \sqrt{16\cos^2 u}\cdot 4\cos u\,du && (1)\\
&= \int_0^{\pi/2} 4\cos u\cdot 4\cos u\,du && (2)\\
&= \int_0^{\pi/2} 16\cos^2 u\,du && (3)\\
&= 16\int_0^{\pi/2} \frac{1+\cos(2u)}{2}\,du && (4)\\
&= 8\int_0^{\pi/2} 1+\cos 2u\,du && (5)\\
&= 8[u+(\sin 2u)/2]_0^{\pi/2} && (6)\\
&= 8\pi/2 = 4\pi && (7)
\end{aligned}$$

2 OR Problem 2

2. Find $\int_{-5/2}^{5/2} \sqrt{25-4x^2}\,dx$

We have the form $\sqrt{a^2-y^2}$ for $a=5$ and $y=2x$. Thus we'll do a sub $y=a\sin u$, so $2x=5\sin u$. Then $2\,dx = 5\cos u\,du \implies dx = 5(\cos u)/2\,du$, so

$$\begin{aligned}
\int_{-5/2}^{5/2} \sqrt{25-4x^2}\,dx &= \int_{-5/2}^{5/2} \sqrt{25-(2x)^2}\,dx && (8)\\
&= \int_{u=\sin^{-1}(2(-5/2)/5)}^{u=\sin^{-1}(2(5/2)/5)} (\sqrt{25-(5\sin u)^2})(5\cos u/2\,du) && (9)\\
&= \frac{5}{2}\int_{u=\sin^{-1}(-1)}^{u=\sin^{-1}(1)} \sqrt{25-25\sin^2 u}\cos u\,du && (10)\\
&= \frac{5}{2}\int_{-\pi/2}^{\pi/2} \sqrt{25\cos^2 u}\cos u\,du && (11)\\
&= 25/2\int_{-\pi/2}^{\pi/2} \cos^2 u\,du && (12)\\
&= 25\int_0^{\pi/2} \cos^2 u\,du && (13)\\
&= 25\int_0^{\pi/2} (1+\cos 2u)/2\,du && (14)\\
&= 25/2[u+(\sin 2u)/2)]_0^{\pi/2} = 25/2\cdot\pi/2 = 25\pi/4 && (15)
\end{aligned}$$

3 OR Problem 3

3. Calculate $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}\,dx$

Since we're dealing with the form $\sqrt{x^2-a^2}$, we'll do a trig sub of the form $x = a\sec u$ with $a = 3$, so $dx = 3\sec u \tan u\,du$.

Then,

$$\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}\,dx = \int_{u=\sec^{-1}(3\sqrt{2}/3)}^{u=\sec^{-1}(6/3)} \frac{1}{(3\sec u)^4\sqrt{(3\sec u)^2-9}} 3\sec u \tan u\,du \tag{16}$$

$$= \int_{u=\sec^{-1}(\sqrt{2})}^{u=\sec^{-1}(2)} \frac{1}{3^4\sec^4 u\sqrt{9\sec^2 u - 9}} 3\sec u \tan u\,du \tag{17}$$

$$= \int_{u=\pi/4}^{u=\pi/3} \frac{1}{3^3\sec^3 u\sqrt{9\tan^2 u}} \tan u\,du \tag{18}$$

$$= \int_{u=\pi/4}^{u=\pi/3} \frac{1}{(3^3\sec^3 u)3\tan u} \tan u\,du \tag{19}$$

$$= \int_{u=\pi/4}^{u=\pi/3} \frac{1}{3^4\sec^3 u}\,du \tag{20}$$

$$= \frac{1}{3^4}\int_{u=\pi/4}^{u=\pi/3} \cos^3 u\,du \tag{21}$$

$$= \frac{1}{3^4}\int_{u=\pi/4}^{u=\pi/3} (1-\sin^2 u)\cos u\,du \tag{22}$$

$$= \frac{1}{3^4}(\int_{u=\pi/4}^{u=\pi/3} \cos u\,du - \int_{u=\pi/4}^{u=\pi/3} \sin^2 u \cos u\,du) \tag{23}$$

$$= \frac{1}{3^4}([\sin u]_{\pi/4}^{\pi/3} - \int w^2\,dw) \tag{24}$$

$$= \frac{1}{3^4}(\sqrt{3}/2 - \sqrt{2}/2 - [(\sin u)^3/3]_{\pi/4}^{\pi/3}) \tag{25}$$

$$= \frac{1}{3^4}(\sqrt{3}/2 - \sqrt{2}/2 - (\sqrt{3}/2)^3/3 + (\sqrt{2}/2)^3/3) \tag{26}$$

$$= \frac{1}{3^4}(3\sqrt{3}/8 - 5\sqrt{2}/12) = \frac{9\sqrt{3}-10\sqrt{2}}{3^4\cdot 24} = \frac{9\sqrt{3}-10\sqrt{2}}{1944} \tag{27}$$

4 OR Problem 4

4. Find $\int \frac{1}{\sqrt{4x^2+4x+2}}\,dx$.

Complete the square in the denominator:

$$4x^2+4x+2 = 4(x^2+x+\frac{1}{2}) = 4(x^2+x+\frac{1}{2}\pm(1/2)^2) = 4(x^2+x+1/4+1/2-1/4) \tag{28}$$

$$= 4((x+1/2)^2+1/4) \tag{29}$$

$$\tag{30}$$

Then, we have $\int \frac{1}{\sqrt{4x^2+4x+2}}\,dx = \int \frac{1}{\sqrt{4((x+1/2)^2+1/4)}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x+1/2)^2+1/4}}\,dx$.

Hence our integral contains $\sqrt{y^2+a^2}$ for $y = x+1/2$, $a = \frac{1}{2}$, so we'll do a $y = a\tan u$ trig sub, i.e. $x + 1/2 = \frac{1}{2}\tan u$. Note $dx = \frac{1}{2}\sec^2 u\,du$.

Then

$$\frac{1}{2}\int \frac{1}{\sqrt{(x+1/2)^2+1/4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(\tan u/2)^2+1/4}}((\sec^2 u)/2\,du) \tag{31}$$

$$= \frac{1}{4}\int \frac{\sec^2 u}{\sqrt{(\tan^2 u)/4+1/4}}\,du \tag{32}$$

$$= \frac{1}{4}\int \frac{\sec^2 u}{\sqrt{(\sec^2 u)/4}}\,du \tag{33}$$

$$= \frac{1}{4}\int \frac{\sec^2 u}{(\sec u)/2}\,du \tag{34}$$

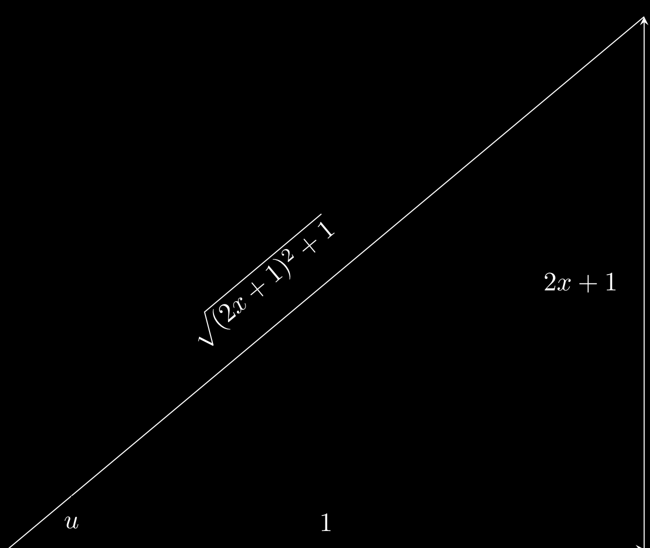
$$= \frac{1}{2}\int \sec u\,du \tag{35}$$

$$= \frac{1}{2}\ln|\sec u+\tan u|+C \tag{36}$$

$$= \frac{1}{2}\ln|\sqrt{(2x+1)^2+1}+2x+1|+C \tag{37}$$

$$= \frac{1}{2}\ln|\sqrt{4x^2+4x+2}+2x+1|+C \tag{38}$$

Where we use our triangle to change from u's back to x's: note $\tan u = 2x+1 = \frac{2x+1}{1}$



Thus $\sec u = \frac{\sqrt{(2x+1)^2+1}}{1} = \sqrt{4x^2+4x+2}$.